Walton MathFest Junior Varsity Test Solutions

Solutions

1. The value of the expression

$$\frac{8^0 + 8^1}{8^{\frac{1}{3}} + 8^{\frac{4}{3}}}$$

can be written as the simplified fraction $\frac{a}{b}$. Find $a + b$.

Directly compute the values to simplify the sum to

$$\frac{1+8}{2+16} = \frac{1}{2} \rightarrow \boxed{\mathbf{3.}}$$

2. Devon has the list of numbers:

$$n-1,\ n,\ n+1,\ n+2,\ n+3,\ n+4,\ n+5$$

for some value of n such that the median of these numbers is 3. What is the mean of these numbers?

The median is 3 which implies that $n+2 = 3 \implies n = 1$. We now know that the numbers are 0,1,. . . ,6, and the average of these is $\boxed{\mathbf{3.}}$

3. If a regular hexagon has a perimeter of 36, the area of the hexagon can be expressed as $a\sqrt{b}$, for positive integers a, b such that the radical is simplified. Find $a + b$.

Recall that the area of an equilateral triangle with sidelength s is

$$A = \frac{s^2\sqrt{3}}{2}.$$

The hexagon consists of six equilateral triangles with sidelength 6, and so the area is

$$6 \cdot \frac{36\sqrt{3}}{4} = 54\sqrt{3} \rightarrow \boxed{\mathbf{57.}}$$

4. In how many different ways can the letters of the word WALTON be arranged?

The number of permutations for 6 letters is simply $6! = \boxed{\mathbf{720.}}$

5. Sally has an average of 82 over 5 math tests. What score does Sally need on her next math test to have an average of 85?

The sum of the scores of Sally's tests is currently $82 \cdot 5 = 410$ and she needs this sum to be $85 \cdot 6 = 510$ in order to have an 85 average over 6 tests. We see that $510 - 410 = \boxed{\mathbf{100.}}$

6. What is the fewest number of lines of symmetry a triangle can have?

We consider a scalene triangle. The answer is $\boxed{\mathbf{0.}}$

7. Sabrina adds three numbers in an arithmetic sequence and gets 12. What was the middle number in this sequence?

Let the three numbers be $a, a+d, a+2d$. We are trying to find $a+d$. We know that

$$a + a + d + a + 2d = 3a + 3d = 3(a+d) = 12 \implies a + d = \boxed{\mathbf{4.}}$$

8. A fair coin is flipped 6 times. The probability that the coin lands heads 5 times, and tails 1 time can be expressed as $\frac{a}{b}$ where a, b are relatively prime positive integers (i.e. the fraction is simplified as much as possible). Find $a + b$.

If the coin lands tails first, and heads the rest of the flips, the probability of this happening is $\frac{1}{2^6} = \frac{1}{64}$. However, the coin can land tails on any of the 6 flips, so we multiply by 6 to get $\frac{6}{64} = \frac{3}{32} \rightarrow \boxed{\mathbf{35.}}$

9. What is the smallest positive integer n such that

$$\frac{121}{12n+4}$$

is reducible?

Notice that $121 = 11^2$. **We see that the fraction is reducible if and only if** $12n+4$ **is divisible by 11. We see that**

$$12n + 4 \equiv n + 4 \equiv 0 \pmod{11}$$

when $n = \boxed{\textbf{7.}}$

10. Louis defines an operation $\star$ such that $m \star n = mn + m + n + 1$. Find

$$\frac{44 \star 23}{8 \star 5}.$$

Computing the value directly is an option, but we can also notice that $m \star n = (m+1)(n+1)$ **and so**

$$\frac{44 \star 23}{8 \star 5} = \frac{45 \cdot 24}{9 \cdot 6} = 5 \cdot 4 = \boxed{\textbf{20.}}$$

11. Let n be a positive integer such that the sum of the first n consecutive odd integers is 900. Find the value of n.

Note that $1 = 1^2, 1 + 3 = 2^2, 1 + 3 + 5 = 3^2$, **and so on, so our answer is** $\boxed{\textbf{30}}$ **because the first** n **consecutive odd integers sum to** n^2. **One can prove this using the arithmetic sum formula or by constructing a square with new terms adding a new layer to the square.**

12. The value of the product

$$\frac{1}{3} \times \frac{2}{4} \times \frac{3}{5} \times \cdots \times \frac{29}{31}$$

can be expressed as $\frac{a}{b}$ where a and b are relatively prime positive integers. Find $a + b$.

Listing out a couple terms, we see that every term in the numerator and denominator cancel out, except for a 2 in the numerator and a $30 \cdot 31$ **in the denominator. We compute**

$$\frac{2}{30 \cdot 31} = \frac{1}{465} \to \boxed{\textbf{466.}}$$

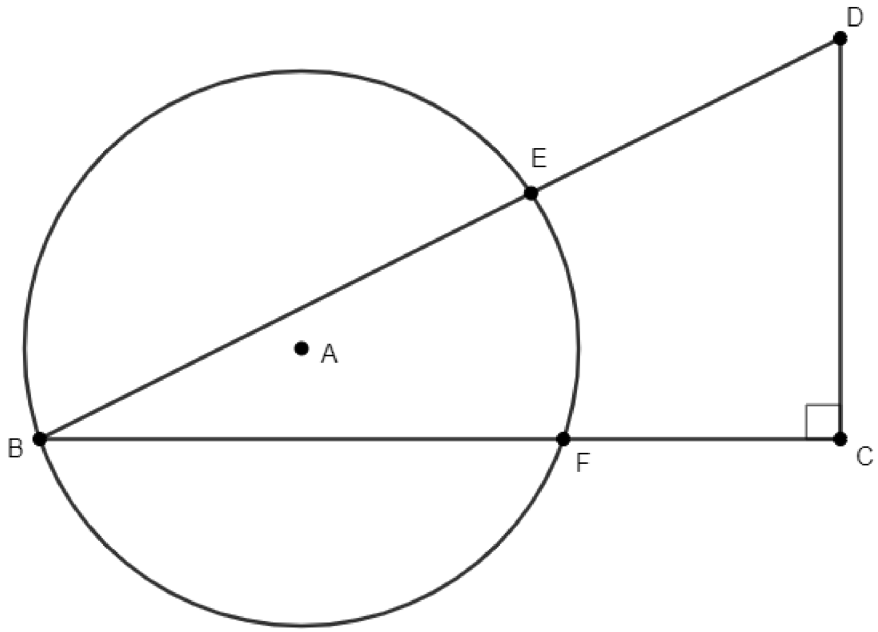
13. A three-digit positive integer $\overline{3MN}$, where M and N represent the tens and ones digit respectively, has the property that it is evenly divisible by 12, and $M + N$ is also evenly divisible by 12. Find the maximum possible value of $10 \cdot M + N$.

Listing out the multiples of 12 between 300 and 399, we see that we are only restricted to

$$300, 312, 324, 336, 348, 360, 372, 384, 396.$$

Note that both 348 and 384 satisfy the condition but the $10 \cdot M + N$ **value is maximized with 384 and so our answer is** $\boxed{\textbf{84.}}$

14. In the figure below, the circle has a radius of 1. The length of the arc EF is $\frac{\pi}{3}$, and $DB = 16$. The length of segment BC can be written as $a\sqrt{b}$ where all radicals are simplified. Find $a + b$.



Because arc $EF = \frac{\pi}{3}$ **and the radius of the circle is 1, we know that** $\angle EBF = \frac{\pi}{6} = 30°$ **and so** BCD **is a** $30° - 60° - 90°$ **triangle. If** $DB = 16$, **then** $DC = \frac{DB}{2} = 8$ **and** $BC = DC\sqrt{3} = 8\sqrt{3} \rightarrow \boxed{\textbf{11.}}$

15. Four circles and one line are drawn in a plane. What is the maximum number of points that are the intersection of two or more figures?

The number of intersections between circles are maximized when each circle intersects any other circle two times. Two circles can intersect at 2 points. Three circles can intersect at 6 points. Four circles can intersect at 12. A line can intersect each of the four circles at most two times, and so our answer is $12 + 4 \cdot 2 = \boxed{\textbf{20.}}$

16. Lily has to divide 12 books into boxes that hold 3 books each. If the boxes are indistinguishable, what is the remainder when the amount of ways the books can be distributed is divided by 10?

The value is

$$\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3}.$$

We could compute this, but notice that 10 is a factor of $\binom{12}{3}$ **and so the units digit will be** $\boxed{\textbf{0.}}$

17. Jacob, Sam, and Dave are each given one positive integer. If Jacob was given a number that is equal to 19 plus the sum of the number of Sam and Dave and the square of Jacob's number is equal to 703 plus the square of the sum of Sam's and Dave's numbers, what is the total sum of numbers of Jacob, Sam, and Dave?

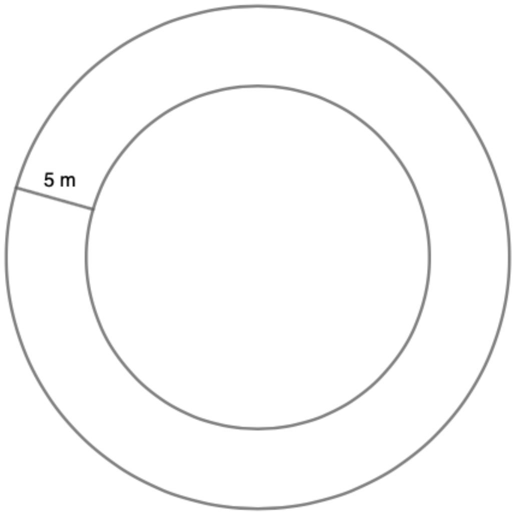
Let j, s, d **represent Jacob, Sam, and Dave's numbers respectively. Let** $k = s + d$ **and we see that**

$$j = k + 19$$
$$j^2 = k^2 + 703.$$

Squaring the first expression and solving the system of equations we see that $k = 9$ **and** $j = 28$. **We now know that**

$$j + k = j + (s + d) = 28 + 9 = \boxed{\textbf{37.}}$$

18. Tim walks around 2 concentric circles at a constant speed of x. The distance between the outer circle and the inner circle is 5 meters. It takes Tim 10 more seconds to walk around the outer circle than the time to walk around the inner circle. Find the integer that is closest to $10x$ when x is expressed in meters per second.



Without loss of generality, we can assume that the inner circle has radius 1 and so the outer circle has radius 6. The inner circle has a diameter of 2 and a circumference of 2π **and the outer circle has a diameter of 12 and a circumference of** 12π. **It takes Tim 10 more seconds to walk a distance that differs by** 10π **and so his speed in meters per second is** π. **We know that**

$$10\pi \approx 10(3.14) \approx \boxed{\mathbf{31.}}$$

19. Luis is attempting to find the last 2 digits of 7^{49}. What is the sum of these digits?

We use Chinese Remainder Theorem after computing that

$$7^{49} \equiv (-1)^{49} \equiv 3 \pmod 4$$

and

$$7^{49} \equiv 7 \cdot (49)^{24} \equiv 7 \cdot (-1)^{24} \equiv 7 \pmod{25}.$$

The only number less than 100 that satisfies these modulos is 07 **so** $7 + 0 = \boxed{\mathbf{7.}}$

20. A circle ω has two chords AB and CD which intersect at E. Given that $DC = 8$ and $AE = 3$, and the measure of CE is an integer, find the sum of all possible integer values of BE.

Let $x = DE$. **Using power of a point, we know that**

$$(x)(8-x) = (3)(CE) \implies CE = \frac{x(8-x)}{3}.$$

Because CE is an integer, $x(8-x)$ **has to be divisible by 3, and this is true when** $x \in \{2, 3, 5, 6\}$. **Testing each of these values we see that** CE **can equal 4 and 5, so our answer is** $4 + 5 = \boxed{9}$

21. Let $f(0) = 0, f(1) = 1$, and

$$f(n) = f(n-1) - f(n-2)$$

for all integers $n \geq 2$. Find the value of

$$f(0) + f(1) + f(2) + \ldots + f(602).$$

Writing out a couple values of the function,

$$[0, 1, 1, 0, -1, -1], [0, 1, 1, 0, -1, -1] \ldots$$

we see that the function is periodic with period 6. Every 6 terms grouped in that manner sum to 0. Each group begins with a value of $f(n)$ such that n is divisible by 6. Our summation simplifies to

$$f(600) + f(601) + f(602) = 0 + 1 + 1 = \boxed{\textbf{2.}}$$

22. Luis is also challenged with finding the number of 0's at the end of 49! (also known as the number of trailing zeros). What is this number?

The number of factors of 10 in the number is equivalent to the number of trailing zeros. The number of factors of 10 is minimum of the number of factors of 5 and the number of factors of 2. We see that there are obviously fewer factors of 5 and so we compute

$$\lfloor \frac{49}{5} \rfloor + \lfloor \frac{49}{25} \rfloor = 9 + 1 = \boxed{\textbf{10.}}$$

23. Jim and Alice play a game. They draw an equilateral triangle of side length 6 and then inscribe a circle inside of it. They then take turns attempting to get a ball to stop in the circle. Assuming the ball's movement is random, what is the probability that Jim fails and then Alice succeeds? The answer can be written in the form

$$\pi \left(\frac{a\sqrt{b} - \pi}{c} \right)$$

where all fractions and radicals are simplified as much as possible. Find $a + b + c$.

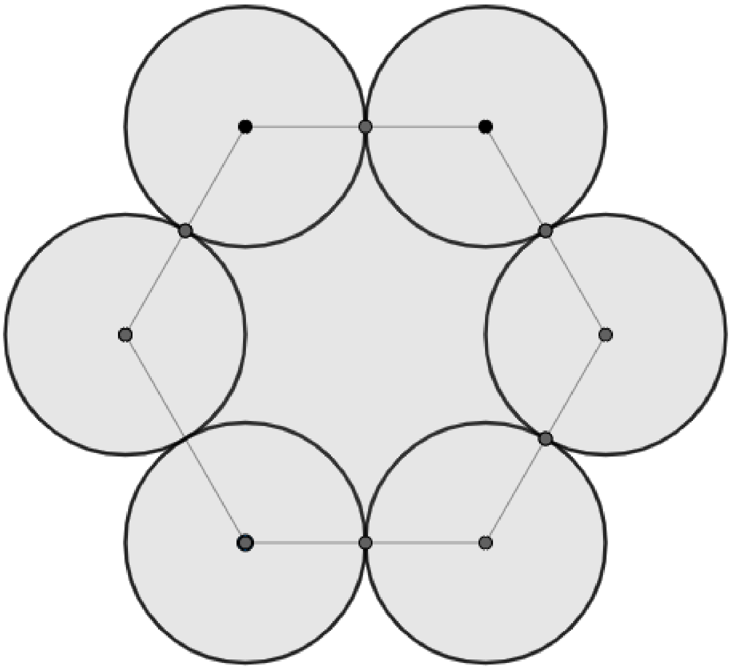
The area of the circle is 3π. The area of the triangle is $9\sqrt{3}$. The probability is thus

$$\frac{3\pi}{9\sqrt{3}} \cdot \frac{9\sqrt{3} - 3\pi}{9\sqrt{3}} = \pi \left(\frac{3\sqrt{3} - \pi}{27} \right) \to 3 + 3 + 27 = \boxed{\textbf{33.}}$$

24. Circles with radius 2 are placed as shown in the figure below. The circles are each tangent to 2 other circles. The area of the shaded region can be expressed as $a\pi + b\sqrt{c}$ where a, b, c are positive integers and the radical is simplified. Find $a + b + c$.

Notice that the shaded region consists of a hexagon with sidelength 4, and the sum of the circular sectors is 4 circles with radius 2. The area is thus

$$6 \cdot \frac{16\sqrt{3}}{4} + 4(2)^2\pi = 16 + 24\sqrt{3} \to \boxed{\textbf{43.}}$$



25. The ratio of the sum of the odd factors and the sum of all factors of 1820 can be represented as $\frac{m}{n}$. Find $m + n$.

Let the sum of the odd factors be k. Notice how we can "tack" on factors of 2. For example, $2k$ is the sum of all even factors of the number that are divisible by 2 but not 4, and similarly for $4k$. Our ratio becomes

$$\frac{k}{k + 2k + 4k} = \frac{k}{7k} = \frac{1}{7} \rightarrow \boxed{\textbf{8.}}$$