Solutions

1. The mean grade of Mr. Mister's basket weaving class is exactly 65. One day, a transfer student comes into the class, and his grade raises the class average to exactly 66. If the transfer student's grade is a 93, how many people were originally in the class?

   The grade sum of the pre-transfer class is $65x$, where x is the number of people in the class. The grade sum of the post-transfer class is $66(x+1)$.

   The grade difference between the two classes is $66x + 66 - 65x = x + 66$, and should be equal to 93.

   Hence,
   $$\mathbf{x = 27}$$

2. In how many different ways can the letters of $WALTON$ be arranged?

   There are 6 places for the first letter, 5 places for the second, 4 for the third and so forth; thus the answer is just
   $$\mathbf{6! = 720}$$
   .

3. In an acute triangle, all three angles are prime numbers. What is the sum of the digits of all the angles in the triangle?

   Since all three numbers add up to 180, and prime numbers are all odd except for 2, 2 has to be one angle. Because the triangle is acute, all angles are less than 90 degrees, so the remaining two angles have to both be 89. Thus, the answer is
   $$\mathbf{2+8+9+8+9=36}$$

4. If a is a real number such that $5^a = 2$, calculate the first digit of the real number y given that $y = 25^{2.5a+2}$.

   Via exponent rules:
   $$y = (25^{2.5a})(25^2)$$
   $$y = 625 * (5^2)^{2.5a}$$
   $$y = 625 * 5^{5a}$$
   $$y = 625 * (5^a)^5$$
   $$y = 625(2^5) = 20000$$

   Thus, the answer is just **2**

5. You can paint a wall in 20 minutes. However, if your friend joins and helps you, you can do the job in 15 minutes. If you were not there, how long would it have taken your friend to paint the wall by himself?

   Your rate of painting the wall is $\frac{1}{20}$ walls per minute, and with you and your friend paints the wall in a rate of $\frac{1}{15}$ walls per minute. Thus, your friend by himself paints at a rate of $\frac{1}{15} - \frac{1}{20} = \frac{1}{60}$ walls per minute, or it would take **60** minutes to paint the wall.

6. The first number of a geometric sequence is 1, and the 17th number of the same geometric sequence is 81. What is the 25th number of the geometric sequence?

   The 17th number is $r^{16} = 81$. We're looking for the 25th number, or
   $$r^{24} = r^{16} * r^8 = 81 * 9 = \mathbf{729}$$

7. Louis defines a function Ω such that $m\Omega n = mn + m + n + 1$. Find the value of $\frac{44\Omega 23}{8\Omega 5}$.

   Notice that the function factorizes as $(m+1)(n+1)$. Thus:
   $$\frac{44\Omega 23}{8\Omega 5} = \frac{45 * 24}{9 * 6} = 5 * 4 = \mathbf{20}$$

8. If n is an odd integer such that the sum of the first consecutive odd numbers up to n(which in other words, is $1+3+5+...(n-2)+n$) is 900, what is n?

   The formula for finding the sum of the first x consecutive odd numbers is just x^2, so x is 30 and thus n is the xth odd number, or $30*2-1=$ **59**.

9. Tim walks around 2 concentric circles at a constant speed of x. The distance between the outer circle and the inner circle is 5 meters. It takes Tim 10 more seconds to walk around the outer circle than the time to walk around the inner circle. Find $10x$ in m/s rounded to the nearest whole number.

   Let the radius of the inner circle be r. The circumference of the inner circle is $2\pi r$, and the circumference of the outer circle is $2\pi(r+5)=2\pi r+10\pi$, so the outer circle is just 10π meters longer, and since it takes Tim 10 seconds to walk 10π more meters, he walks π meters per second, or **31** meters per second.

10. When graphed on a normal xy plane, $z*\pi$, where z is an integer, represents the area enclosed by all the coordinates (a,b) that satisfy the condition that the following polynomial contains only one real root:

$$f(x)=ax^2+2(a+b-10)x+2b$$

   Find the value of z.

   For a quadratic to have only one real root, its Discriminant (D) must be equal to 0.

$$D=b^2-4ac=4(a+b-10)^2-4(a)(2b)=0$$
$$(a+b-10)^2-2ab=0$$
$$(a^2+b^2+100+2ab-20a-20b)-2ab=0$$
$$a^2-20a+b^2-20b+100=0$$
$$(a-10)^2-100+(b-10)^2-100+100=0$$
$$(a-10)^2+(b-10)^2-100=0$$
$$(a-10)^2+(b-10)^2=10^2$$

   This is the equation giving the values of a and b that satisfies the condition of the polynomial, and is clearly shown to enclose a circle with a radius of 10 and center (10,10) when plotted on a xy graph. The area of the circle is $\pi*r^2=\pi*10^2=100\pi$.
   So $z*\pi=100\pi$, and thus:

$$\mathbf{z=100}$$

11. How many 5 digit palindromic numbers (numbers which remain the same reversed; for example, 25652 is palindromic because the reverse of 25652 is 25652) are there which are divisible by 3?

   Since it's a palindrome, we only have to care about the first three digits; the tens and units digit have to be equal to the first two digits.

   Doing a little casework on whether or not the middle digit is divisible by 3 reveals that no matter whether or not the middle digit leaves a remainder of 0, 1, or 2 when being divided by 3, there are always 30 choices for the other digits. The answer is just thus

$$30*10=\mathbf{900}$$

12. In the regular hexagon $ABCDEF$, triangle ABC has area $9\sqrt{3}$. The area of the quadrilateral $ABCD$ is in the form $a\sqrt{b}$, where the radical ($\sqrt{b}$) is as simplified as possible. Find $a+b$.

   Solve for the side length of the hexagon using Pythagorean's theorem. The side length is 6. Then just solve for area of triangle ACD, which is

$$18\sqrt{3}$$

   Therefore, total area is

$$18\sqrt{3}+9\sqrt{3}=\mathbf{27\sqrt{3}}$$

13. Let $f(0) = 0$, $f(1) = 1$, and $f(n) = f(n-1) - f(n-2)$ for all integers $n >= 2$. Find the value of $f(0) + f(1) + f(2) + \ldots + f(602)$.

    Trying out a couple of numbers of the sequence we find that the sequence goes:

    $$0, 1, 1, 0, -1, -1, 0, 1, 1...$$

    There's a pattern of $0, 1, 1, 0, -1, -1$ that repeats in groups of 6, which has a sum of 0. Since $f(602)$ is the 603rd number of the sequence, all the sums add up to 0 except for $f(600), f(601), f(602)$ which is the first three numbers of the pattern: 0, 1, 1. Therefore, the answer is just **2**

14. The numbers a, b, c are non-negative integers. How many possible pairs (a, b, c) satisfy the equation $a+b+c = 9$?

    This is just stars and bars, so the answer is just

    $$\binom{11}{2} = \mathbf{55}$$

15. A laser shot from $(3, 5)$ bounces off the x axis then the y axis and hits a target at $(3, 3)$. The point at which the laser bounced off the x-axis can be expressed as $(\frac{a}{b}, 0)$, where a and b are relatively prime, positive integers. What is $a + b$?

    We can reflect the target across the y and x axis in order to get to the point where the laser would hit originally without being reflected (this is true because then we reflect the laser across the x axis then the y axis to get the target again), so the new target becomes $(-3, -3)$. The line between $(3, 5)$ and $(-3, -3)$ can be solved to be $\frac{4x}{3} + 1$. At $y = 0$, $x = -\frac{3}{4}$. As such, $a = 3$ and $b = 4$, so the solution is **7**

16. The volume of a regular triangular pyramid (regular meaning that all side lengths are the same) with side length 6 is in the form $a\sqrt{b}$, where the radical($\sqrt{b}$) is as simplified as possible. Find $a + b$.

    First, we find the height. The length from a vertex on the base of the pyramid to the point directly beneath the apex is equal to the length of a vertex to the circumcenter of the base. Since the base is an equilateral triangle, this is equal to $2\sqrt{3}$, and thus the height is $\sqrt{36-12} = \sqrt{24} = 2\sqrt{6}$. Therefore, the volume is

    $$\frac{1}{3}2\sqrt{6} * \frac{6^2\sqrt{3}}{4} = 18\sqrt{2}$$

    .

    Hence, the answer is just **20**.

17. How many pairs of positive integers (x, y, z) satisfy the equation $x + xy + xyz + 4 = 2021$?

    $$x + xy + xyz = 2017$$
    $$x(1 + y + yz) = 2017$$

    Since 2017 is a prime number and y and z are both positive integers, x must be equal to 1. This gives:

    $$1 + y + yz = 2017$$
    $$y + yz = 2016$$
    $$y(1 + z) = 2016$$

    y must equal all divisors of 2016 except for 2016 itself because having y as 2016 would make z equal 0, which does not satisfy the condition of all positive integers. Therefore, the (number of divisors of 2016-1) gives the the number values of y that satisfy the equation and therefore the number of (x,y,z) pairs.

    2016 Number of divisors:

    2016 Prime Factorization:

    $$(2^5)(3^2)(7)$$

    # of divisors =

    $$(5 + 1)(2 + 1)(1 + 1) = 36$$

    Number of (x,y,z) pairs = 36-1= **35**

18. A 12-sided regular polygon (dodecagon) and a square are inscribed in a circle of radius 1. The ratio between the area of the dodecagon to the area of the square can be expressed as $\frac{a}{b}$ where a and b are relatively prime, positive integers. What is $a+b$?

    Area of a dodecagon slice $= \frac{1}{2}absinC$, where A and B are the lengths of the equal sides. $\frac{1}{2}*1*sin(30)=\frac{1}{4}$, and there are 12 slices so the area of the dodecagon is 2. Area of square given a diagonal of 2 (the diameter) is also just 2. 3/2 is our fraction, and $3+2=\mathbf{5}$

19. Let set A contain all the natural numbers from 1 to 100. Let B be the subset of A such that each element in B satisfies the following condition: the LCM of the element and 100 is equal to 100 times the element. How many elements are in B?

    LCM * GCD = Product of two numbers. If LCM = product, GCD = 1

    Euler's Totient: $100*(1-\frac{1}{2})*(1-\frac{1}{5})=40$

    Answer: **40**

20. What are the last two digits of $43^{43^{43}}$?

    Via modulos and Chinese-Remainder theorem:

    $$43^{43^{43}} mod4 \equiv (-1)^{odd} mod4 \equiv -1 mod4$$

    $$43^{43^{43}} mod25 \equiv 18^{43^{43}} \equiv 324^{(43^{43}-1)/2} * 18 \equiv (-1)^{(43^{43}-1)/2} * 18$$

    $$43^{43} \equiv -1 \equiv 3mod4$$

    so

    $$(-1)^{(43^{43}-1)/2} * 18 \equiv (-1)^{odd} * 18 \equiv -18 \equiv 7mod25$$

    Putting it together, we get that the answer is

    $$7mod100 = \mathbf{7}$$

21. Andy rolls six fair normal dice all at once at a carnival game. If the lowest number he rolls out on any of the six dice that were tossed is 5, then he gets a cash prize of \$10 dollars. Let $\frac{a}{b}$ be the fraction equivalent to the probability Andy can earn a cash prize from the carnival game. Find the value of a.

    To gain a cash prize, Andy must roll a dice that is exactly 5 and is the lowest number out of all of the six dice.
    Probability of 6 Dice Rolls being > 5 : $(2/6)^6$
    Probability of 6 Dice rolls being 6 : $(\frac{1}{6})^6$
    We need to find the probability of the 6 Dice Rolls being at least 5, but not all dice rolls being 6 to ensure at least one dice roll being the number 5 as the least value among all the dice rolls:

    $$P = (2/6)^6 - (\frac{1}{6})^6 = (2^6-1)/(6^6)$$

    $$a = (2^6-1) = \mathbf{63}$$

22. $F(x)$ is a quartic polynomial under the equation

    $$F(x) = x^4 + 2020x^3 - 2019x^2 - 2018x + 2017$$

    $G(x)$ is a similar quartic polynomial, containing a coefficient of 1 for x^4 just as $F(x)$ does; however, it contains real roots that are the perfect squares of the roots of $F(x)$. $H(x)$ is equal to $\frac{G(x^2)}{F(x)}$ and is a quartic polynomial like $F(x)$ and $G(x)$. Calculate $|H(1)|$.

    Suppose the four roots of F(x) are a, b, c, d, making

    $$F(x) = (x-a)(x-b)(x-c)(x-d)$$

Because the roots of G(x) are squares of F(x)'s roots,

$$G(x) = (x - a^2)(x - b^2)(x - c^2)(x - d^2)$$

$$G(x^2) = (x^2 - a^2)(x^2 - b^2)(x^2x - c^2)(x^2 - d^2) = (x + a)(x - a)(x + b)(x - b)(x + c)(x - c)(x + d)(x - d)$$

H(x) is equal $\frac{G(x^2)}{F(x)}$, so

$$H(x) = [(x + a)(x - a)(x + b)(x - b)(x + c)(x - c)(x + d)(x - d)]/[(x - a)(x - b)(x - c)(x - d)]$$

$$H(x) = (x + a)(x + b)(x + c)(x + d) = (-x - a)(-x - b)(x - c)(x - d)$$

$$F(-x) = (-x - a)(-x - b)(x - c)(x - d)$$

Therefore,

$$H(x) = F(-x)$$

$$H(1) = F(-1)$$

$$F(-1) = (-1)^4 + 2020(-1)^3 - 2019(-1)^2 - 2018(-1) + 2017$$

$$F(-1) = 1 - 2020 - 2019 + 2018 + 2017$$

$$F(-1) = -3$$

Thus, the answer is

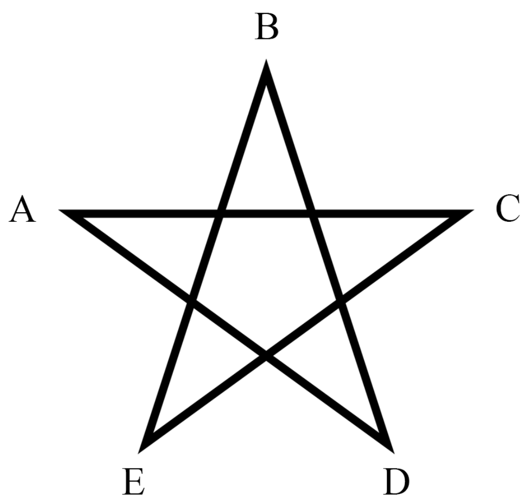
$$|-3| = \mathbf{3}$$

23. A pentagram (five-pointed star) has five magical torches placed at each of its vertices. The torches have the following properties:

(a) A torch can either be lit or unlit.

(b) Torches can be lit manually through an action, but cannot be put out manually.

(c) If a torch is lit manually through an action, the two torches along its adjacent vertices have their states toggled.
For example, if only torch C is initially lit and then torch A is manually lit through an action, only torches A and D will be lit after this action.



What is the maximum number of actions needed to light all of the torches on the pentagram, given any initial condition?

(Apologies in advance for non-latex text - these diagrams were hard to do in latex!)

The reasoning behind the solution is to start with the final solution, where all torches are lit, and work your way backwards to find all the possible combinations of torch lighting for the initial setup.

○○○○○

If you toggle a white to a black the surrounding ones also turn black. The last one loops around to the first and vice versa.

The final solution is this:
●●●●●
Which can only be made by toggling this:
●○○○● (equivalent to ○●●●○)
  ^
Which can only be made by toggling this (i'll just continue)
○○○●○ 2
    ^
○○●○● 3
   ^
○●○●● 4
  ^
●○●●● 5 (equivalent to ●●●●○)

●●○○● 6 (equivalent to ○●●●○)
  ^
○○●○● (same as 3)

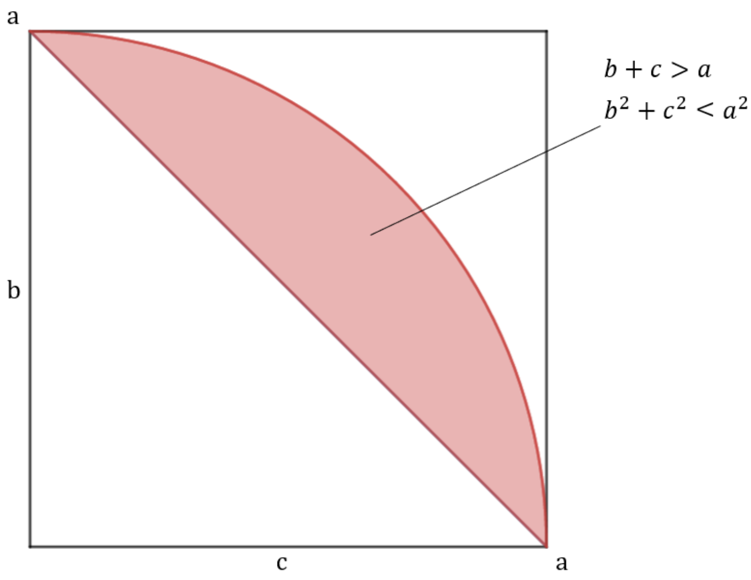
There are other permutations that could result in those positions but I only chose the ones that were unique, to try to get all possible positions that lead to a solution. If a position can be made by shifting all of the dots over by one (equivalent to rotating the pentagram), it's identical. These are the only possible colorings of the pentagon.

Finally, we have one more case - the completely empty set of torches case. The case is: 00000 to 01110 to 11101 to 11010 to 10100 to 01000 to 00110 to 11111.

Therefore, the maximum number of actions needed to light all of the torches is **7**.

24. Three numbers are chosen randomly between 0 and 1. The probability that the three numbers make up the side lengths of an obtuse triangle is in the form $\frac{\pi}{a} - \frac{b}{c}$ where $\frac{b}{c}$ is a fraction as simplified as possible, and $\frac{\pi}{a}$ is as simplified as possible. Find $a + b + c$.

    Let the three numbers chosen be denoted as a,b and c, and have $a \geq b \geq c$. Then, because we want $a < b + c$)(because triangle inequality) and $a^2 > b^2 + c^2$(because the triangle is obtuse), we have the following diagram:



The red space is what we're looking for, which is

$$(\frac{\pi a^2}{4} - \frac{a^2}{2})/a^2 = (\pi/4 - 1/2)$$

so the answer is $4 + 1 + 2 =$ **7**

25. The positive integers from 1 to 13 are each assigned a color. If the positive integers x, y, and z follow both $x + y = z$ and x, y, and z are all assigned the same color, call this a "monochromatic" solution to $x + y = z$. In addition, x, y, and z do not have to be distinct. What is the minimum number of colors needed to color all integers from 1 to 13 while ensuring no monochromatic solutions exist?

By trying out combinations, we can smartly and semi-exhaustively try out the number of colors we use starting from 1.

1 clearly doesn't work. 2 doesn't work either, because trying colors out starting from 1, we come into a problem with finding the color for 5.

So does 3 work?

With some trial and error, and a smart method of only assigning the third color when it's desperately needed we get that there does exist a combination, and it is:
a b b a c c c c c a b b a
(where a is the first color, b is the second color, and c is the third color, and the placement corresponds to the number with that color)

Thus, the answer is **3**